AAPT Studies in Pedagogy 1 (2015): Practices in Pedagogy, pp. 201-218
doi: 10.5840/aaptstudies201510114

[This typescript is the same as the published work, with the exception of the style of the notes.]

Teaching the PARC System of Natural Deduction

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Abstract: PARC is an "appended numeral" system of natural deduction that I learned as an undergraduate and have taught for many years. Despite its considerable pedagogical strengths, PARC appears to have never been published. The system features explicit "tracking" of premises and assumptions throughout a derivation, the collapsing of indirect proofs into conditional proofs, and a very simple set of quantificational rules without the long list of exceptions that bedevil students learning existential instantiation and universal generalization. The system can be used with any Copi-style set of inference rules so it is quite adaptable to many mainstream symbolic logic textbooks. Consequently, PARC may be especially attractive to logic teachers who find Jaśkowski/Gentzen-style introduction/elimination rules to be far less "natural" than Copi-style rules. The PARC system is also keyboard-friendly in comparison to the widely adopted Jaśkowski-style graphical subproof system of natural deduction, viz., Fitch diagrams and Copi "bent arrow" diagrams.

The pedagogy of most contemporary symbolic logic textbooks is firmly rooted in the natural deduction systems of Stanisław Jaśkowski[1] and Gerhard Gentzen.[2] First-order logic can be also be taught as an axiomatic system after Frege, Russell, and Hilbert, or using a Gentzen sequent calculus, but this is rare in undergraduate textbooks. Unhappily, the natural deduction systems developed and employed in logic textbooks over the past 80 years vary widely in what is considered "natural."[3] Symbolic logic teachers who teach natural deduction thus continue to face substantial pedagogical choices and challenges.

In what follows, I describe a system of natural deduction that I learned as an undergraduate and have taught for many years. I call the system "PARC," an initialism for the four deduction metarules of sentential and predicate logic, P, A, R, and C. PARC appears to date from the mid-1960s and is clearly derived in part from Patrick Suppes' classic 1957 textbook, *Introduction to Logic*,[4] although there are substantial differences. Other central aspects of PARC appear to be derived from the third edition of Copi's *Symbolic Logic*.[5] A number of textbooks[6]

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[1]Stanisław Jaśkowski, "On the Rules of Suppositions in Formal Logic," *Studia Logica* 1 (1934), 5-32, reprinted in *Polish Logic, 1920—1939,* ed. Storrs McCall (London: Oxford University Press, 1967), 232-258.

[2]Gerhard Gentzen, "Untersuchungen über das Logische Schließen" ["Investigations into Logical Deduction"], *Mathematische Zeitschrift* 39 (1934/5): 176-210, 405-431, in *The Collected Works of Gerhard Gentzen*, trans. M. E. Szabo (Amsterdam: North-Holland Publishing Co., 1969), 68-131.

[3]For example, see Irving Anellis, "Forty Years of 'Unnatural' Natural Deduction and Quantification: A History of First-Order Systems of Natural Deduction from Gentzen to Copi," *Modern Logic* 2 (1991), no. 2: 113–152, and Francis Pelletier, "A Brief History of Natural Deduction," *History and Philosophy of Logic* 20 (1999): 1-31. These papers are essential reading for anyone interested in the history of natural deduction and my research into PARC's origins owes much to them.

[4]Patrick Suppes, *Introduction to Logic* (New York: Van Nostrand Reinhold Co., 1957).

[5]Irving M. Copi, *Symbolic Logic*, 3d ed. (New York: Macmillan Publishing Co, 1967).

[6]Early followers of Suppes include E. J. Lemmon, *Beginning Logic* (London: Thomas Nelson and Sons, 1965), Benson Mates, *Elementary Logic* (New York: Oxford University Press, 1965), and John L. Pollock, *An*

in the 1960s appear to have based their natural deduction systems on Suppes. I will compare PARC with those systems as well as with Gentzen-style introduction-elimination ("int-elim") systems popularized by Fitch[7] and the Copi-style systems that remain popular today.

I hope to encourage logic instructors to consider using PARC in their introductory symbolic logic courses. First, younger faculty are likely to be unfamiliar with PARC's Suppes-style system of premise numbers. The system has considerable pedagogical strengths, compared to current variants of Jaśkowski's graphical subproof system of natural deduction, viz., the so-called Fitch diagrams and Copi's "bent arrow" diagrams found in the majority of current introductory symbolic logic textbooks. Both Fitch's widely-adopted system of vertical and horizontal lines and Copi's bent arrow subproof diagrams are essentially Jaśkowski's system of nested boxes around subproofs with the tops and right sides of the boxes removed.

Second, few contemporary symbolic logic textbooks adequately explain why indirect proof, existential instantiation or elimination—and in some systems, universal generalization or introduction—are all species of conditional proof. Int-elim systems are often the worst offenders here because the subproofs involved in conditional proofs (conditional introduction), indirect proofs (negation introduction and elimination), existential elimination, and universal introduction are all presented as unrelated primitive rules. Copi-style systems often present indirect proof as a primitive proof strategy unrelated to conditional proof.[8]

Third, quantification rules are well-known for their complexity and consequent difficulty for students to master. The history of quantification rules in natural deduction is marked by errors, and some errors have persisted through multiple editions of textbooks.[9] Most errors surround the restrictions on existential instantiation or elimination and universal generalization or introduction. In contrast, PARC's quantification rules are very simple, in part because there is no EI. This solves the philosophical puzzle of how we can allow a line in a proof (by EI) that is not logically implied by any lines in the proof. Instead, the conditional subproof that underlies all EI/∃E rules is explicitly invoked in PARC. The semantical mess of "ambiguous names," "quasivariables," name/variable "flagging," etc., is exposed and simplified, if not avoided entirely.

This paper focuses on these central problems in teaching symbolic logic and how the PARC system provides a genuine alternative to the natural deduction systems in contemporary symbolic logic textbooks.

Introduction to Symbolic Logic (New York: Holt, Rinehart and Winston, 1969). I hope to place the PARC system historically as best as possible in the future, but its precise origin currently remains unknown to me.

[7]Frederick Fitch, *Symbolic Logic: An Introduction* (New York: The Ronald Press, 1952).

[8]Readers accustomed to indirect proofs via a primitive rule may want to skip ahead to Section IV below to see the the underlying conditional proof basis.

[9]Copi's own first and second editions of *Symbolic Logic* are notorious for their quantificational rule errors, as I learned from my logic professor in graduate school, Hugues Leblanc. The history of the debate is examined in Anellis, "Forty Years," 127-145. Seemingly small, but fatal omissions in UG and EG rules persisted in *Logic and Philosophy* through the 10th edition. See Alan Hausman,, Howard Kahane, and Paul Tidman, *Logic and Philosophy: A Modern Introduction*, 12th ed. (Boston: Wadsworth, Cengage Learning, 2013), 272.

I. PARC Overview

In the PARC system of deduction, each line of a proof, derivation or deduction[10] is written down by one of the four rules, P, A, R, or C. Rule P allows any formula whatever to be written down in a proof, while Rules A and R allow lines to be written that are tautologically implied by, or are tautologically equivalent to, previous lines of the deduction. Rule C, for conditional proof, is a standard primitive rule in systems of natural deduction. Here are the PARC deductive rules and definitions. (Quantifier rules are presented later.)

Rule P: Any formula F may be written down as the nth line of a deduction (derivation) if the numeral n is prefixed[11] to it.

Rule A: A formula F may be written down as a line of a deduction if both

(a) there are previous lines of the deduction from which F follows by some elementary argument form (e.g., Copi's Rules of Inference), and

(b) each numeral prefixed to any of these previous lines is prefixed to F.

Rule R: A formula F may be written down as a line of a deduction if both

(a) there is a previous line L of the deduction and a pair of elementary logically equivalent formulae G and H (e.g., Copi's Rules of Equivalence) such that G is a formula that occurs in L and it is possible to transform L into F by putting H in place of one occurrence of G, and

(b) all numerals prefixed to L are prefixed to F.

Rule C: A formula $P \supset Q$ may be written down as a line of a deduction if both

(a) there are previous lines of the deduction P, and Q, such that P was written down by Rule P, and

(b) all numerals prefixed to Q except the numeral prefixed to P are prefixed to $P \supset Q$.

VALIDITY

An argument with premises P1, . . ., Pn and conclusion C is valid if there is a deduction of C as the conclusion from the premises P1, . . ., Pn such that

(a) every line of the deduction is written down by one of the above four Rules, and

(b) each of the premises P1, . . ., Pn is written down by Rule P, and

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[10] I use the terms "derivation," "deduction," and "proof" more or less interchangeably throughout this paper. This excludes Smullyan truth trees, which I do not regard as deductions, per se, following Richard C. Jeffrey, *Formal Logic: Its Scope and Limits* (New York: McGraw-Hill, 1967), 92.

[11] My original PARC notes use the term "appended" when referring to line numbers written down by Rule P. "Prefixed" is more descriptive when referring to premise numbers, since the four Rules, not the premise numbers, are appended to the right end of the line.

(c) every numeral prefixed to the conclusion C is prefixed to one of the premises P1, . . ., Pn.

A formula C is a tautology if there is a deduction of C from zero premises.

II. PARC Rule P and Prefixed Numerals

Let's now consider each of these four PARC rules in turn, beginning with Rule P. Rule P states that "any formula F may be written down as the nth line of a deduction (derivation) if the numeral n is prefixed to it." Like several other first-order natural deduction systems in the 1960s, PARC most likely follows Suppes' 1957 Rule P and premise-numbers, although Quine's 1950 Rule P precedes Suppes:

> Quine: "Rule of premises (P)" – "We may set down any schema as a line at any stage in the course of the deduction, provided that we initiate a new innermost column of stars at that point."[12]
>
> Suppes: "Rule P" – "We may introduce a premise at any point in a derivation."[13]

Rule P is significant for both logical and pedagogical reasons. First, the concept of assumption or supposition lies at the heart of natural deduction. In fact, Jaśkowski introduces a supposition operator, 'S', that is placed in front of a statement that is assumed in a line of a deduction. It is the idea of "suppositional" proof as opposed to the older axiomatic method of logic that Jaśkowski credits to his teacher, Jan Łukasiewicz.[14] As Anellis observes, Gentzen's system of natural deduction shares the same feature of supposition: "[p]roofs are begun with assumptions and the consequences of those assumptions are obtained by discharging the assumptions by conditionalization."[15] It is this model of ordinary mathematical reasoning as opposed to the "formalization of logical deduction . . . developed by Frege, Russell, and Hilbert"[16] that Gentzen is noted for in the history of natural deduction.

Second, when we introduce our students to the concept of a derivation or proof of a conclusion from a set of premises, we emphasize the concept that *every* line of a proof must have a justification for being written down in the column of numbered lines that comprise the proof. This guarantees that every line is logically implied by itself or a subset of the premises. Rule P satisfies this requirement, but many current textbooks do not provide such a rule. For example, my preferred symbolic logic textbook in recent years—Hausman, Kahane, and Tidman's venerable *Logic and Philosophy*—simply says in a footnote that "the letter p . . . [is used to the right of the line] to indicate an argument's premises."[17] The convention is not presented as a syntactic requirement, but rather as an editorial mark.

Third, the relationship between a premise and a temporary assumption can be difficult for students. Hausman provides an "Assumed Premise" rule for conditional and indirect proofs, but does not discuss the similarity between "original" premises and "assumed" premises. In fact,

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[12]Quine, *Methods of Logic*, 157.
[13]Suppes, *Introduction to Logic*, 28.
[14]Anellis, "40 Years," 117.
[15]Ibid.
[16]Gentzen, "Investigations," 68.
[17]Hausman, *Logic and Philosophy*, 89.

this gloss is typical in current introductory first-order logic textbooks and creates unnecessary pedagogical problems. Rule P illuminates the connection in a very direct way, viz., that there is no logical distinction between the premises of an argument and the temporary assumptions that are used in constructing conditional and *reductio ad absurdum* proofs. In both cases, we are simply entitled to assert that *if* these antecedent assumptions are true, *then* this consequent statement is true as well. Systems that have an assumption rule like PARC's Rule P thus make understanding the conditionalization rule—Rule C above—much more accessible to the beginning student. Assumption and conditionalization are interdependent concepts, of course, and making their connection explicit is very useful in the classroom.

Finally, Rule P sometimes strikes students as strange. Such a broad permission to write down "any formula F" seems contrary to the idea of constructing a derivation according to a very limited and precisely defined set of rules. I approach the puzzlement this way with my students: the limitation attached to Rule P is that there is no "free lunch" in a deduction. When you use a previous line in constructing a new line of the deduction, you must cite those source lines—not only by the line numbers themselves, but in terms of the original lines written down by Rule P. Rule P, together with Suppes-style premise-numbering, identifies precisely which premises and/or temporary assumptions logically imply a given line of the proof. Students quickly learn that this feature provides a great advantage both in constructing and understanding a proof.

Here's an example of PARC's rule citation and the prefixed numerals (without any subproofs):

	(1)	$A \supset B$	Rule P	
	(2)	$B \supset C$	Rule P	
	(3)	$(A \supset C) \supset (B \supset D)$	Rule P	
	(4)	$(A \supset D) \supset E$	Rule P	$/\therefore E$
(1,2)	5.	$A \supset C$	Rule A, 1, 2, Hypothetical Syllogism	
(1,2,3)	6.	$B \supset D$	Rule A, 3, 5, Modus Ponens	
(1,2,3)	7.	$A \supset D$	Rule A, 1, 6, Hypothetical Syllogism	
(1,2,3,4)	8.	E	Rule A, 4, 7, Modus Ponens	

Rule P guarantees that each line of a valid deduction is logically implied by the line(s) referenced in the list of prefixed line numbers. In the above proof, lines 1-4 are the logical consequences of themselves, respectively, while line 6 is the logical consequence of premises 1, 2, and 3. It would be a mistake to list line 5 as a premise number of line 6, of course. This distinguishes the justifications that are standardly written to the right of each line of a deduction and the premise numbers listed on the left. Line 8, the desired conclusion, is the logical consequence of lines 1, 2, 3, and 4.[18]

In Suppes's notation, the above proof would look like this, where "T" is Tautological Implication:

{1}	1.	$A \supset B$	P
{2}	2.	$B \supset C$	P
{3}	3.	$(A \supset C) \supset (B \supset D)$	P

[18] I learned the PARC system of prefixed numerals using circles and ovals rather than parentheses around the numerals. I use circles and ovals in lecture as a model for student homework and exams. Students who submit their homework as a word processor document need to use parentheses.

{4}	4.	$(A \supset D) \supset E$	P	$/\therefore E$
{1,2}	5.	$A \supset C$	1, 2, T	
{1,2,3}	6.	$B \supset D$	3, 5, T	
{1,2,3}	7.	$A \supset D$	1, 6, T	
{1,2,3,4}	8.	E	4, 7, T	

As can be seen in comparing the two proof styles above, the advantage of the PARC system of parenthesizing or circling line numbers that are written down by Rule P is that they stand out visually. This is especially useful to students when learning conditional and indirect proofs. However, it should be noted that parenthesizing Rule P line numbers breaks part of the logical basis of the premise-number system as originally conceived by Suppes. Suppes' typography uses the curly braces of set notation to enclose the premise numbers: "[t]his additional notation [the braces] makes it clearer that a given line is a logical consequence of the *set* of premises corresponding to the *set* of numbers attached to the line."[19] In the case of Rule P lines, each assumption is a logical consequence of the set that contains itself. So, both notations have their strengths and weaknesses.

Since derivation lines written down by Rule P must be referenced by any subsequent lines that are tautologically implied by any conjunction of premises of which that line is a conjunct, we can see the value of premise-number notational system over the current, almost universal, Jaśkowski/Fitch/Copi graphical method. The reason is that while notations in introductory logic texts require citing inference rules at the right end of each line of the proof, only the Rule P premise-number system allows the reader to quickly identify the premises or temporary assumptions of the argument that tautologically imply the given line, regardless of whether the proof is direct, conditional, or indirect (reductio ad absurdum). For students learning to construct proofs, this feature provides a ready source of a standard proof strategy hints, e.g., "What premises haven't been employed?" or "What temporary assumptions haven't been discharged?" In my experience, this is a great advantage for beginning logic students.

III. PARC Rules A and R

PARC Rules A (implication rules) and R (equivalence rules) have standard counterparts in current symbolic logic textbooks. Approaches to derivation rules may be roughly sorted into three groups: (1) Gentzen-style int-elim systems that contain only rules that introduce or eliminate logical connectives, (2) systems that have defined sets of implication and equivalence rules, and (3) systems that allow deductions to cite any logical implication or logical equivalence whatever as a justification for a given line of a deduction.

PARC is clearly in category 2. Here is the language of the PARC rules A (implication) and R (logical equivalence or "replacement"):

> PARC Rule A: A formula F may be written down as a line of a deduction if both (a) there are previous lines of the deduction from which F follows by some elementary argument form (e.g., Copi's Rules of Inference), and (b) each numeral prefixed to any of these previous lines is prefixed to F.

[19] Suppes, *Introduction to Logic*, 27-28.

> PARC Rule R: A formula F may be written down as a line of a deduction if both (a) there is a previous line L of the deduction and a pair of elementary logically equivalent formulae G and H (e.g., Copi's Rules of Equivalence) such that G is a formula that occurs in L and it is possible to transform L into F by putting H in place of one occurrence of G, and (b) all numerals prefixed to L are prefixed to F.

The first type of inference/equivalence rules found in int-elim systems are problematic in my experience. While I prefer an int-elim system in my metalogic course, many of us who teach first-order symbolic logic find int-elim systems not to be very "natural." Int-elim systems are often featured in upper-level textbooks that progress quickly to the issues of completeness and soundness, higher-order logics, modal logic, etc. As Fitch points out, "the whole system can most easily be shown to be consistent if no proofs but int-elim proofs are used."[20] Int-elim systems in introductory textbooks typically discuss "derived rules of inference" such as disjunctive syllogism or De Morgan's Theorem to show that traditional rules of inference are guaranteed in the system.[21] This relegation of traditional—read "natural"—rules of inference and equivalence to "derived" status is undeniably attractive in advanced logic courses where the "naturalness" of the deductive system is secondary to ease and clarity in proving soundness and completeness. Leblanc and Wisdom's *Deductive Logic*[22] is a particularly elegant example of this pedagogical choice.

The second sort of natural deduction systems have defined sets of inference and equivalence rules. This approach is very common in contemporary introductory symbolic logic textbooks. One need only look on the inside front and back book covers for lists of deductive rules. These lists typically contain about ten inference rules and eight or nine equivalence rules. Copi's *Symbolic Logic* is arguably the model for this approach[23] and is widely imitated. On the other hand, minimalist rule sets—beyond int-elim systems--are infrequent and require critical axioms. For example, Quine's 1940 system relies only on *modus ponens*. Quine cites Tarski[24] as the source of this ultimately impoverished set of inference rules.[25] Small sets of inference rules mean longer proofs and in many cases, axiom sets. This is clearly not a popular choice in the logic textbook market.

The third sort of natural deduction system has an indefinite set of inference and equivalence rules. For example, in *Methods of Logic*, Quine moves from the minimalist rule set in *Mathematical Logic* to a system at the other end of the spectrum with the following "TF" rule:

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[20]Fitch, *Symbolic Logic*, 31.

[21]That alone won't stop students from complaining that the inordinately long proofs typical of introduction-elimination systems are evidence of a defective system!

[22]Hugues Leblanc and William A. Wisdom, *Deductive Logic*, 3d ed. (Englewood Cliffs, N.J.: Prentice-Hall, [1972, 1976] 1993).

[23]For example, Copi's *Symbolic Logic* has standardly contained the following "Rules of Inference:" Modus Ponens, Modus Tollens, Hypothetical Syllogism, Disjunctive Syllogism, Constructive Dilemma, Destructive Dilemma, Simplification, Conjunction, and Addition; and the following "Rules of Replacement:" De Morgan's Theorem, Commutation, Association, Distribution, Double Negation, Transposition, Material Implication, Material Equivalence, Exportation, and Tautology, in Copi, *Symbolic Logic*, inside back cover.

[24]Alfred Tarski, "Grundzüge des Systemkalkül, Erster Teil," *Fundamenta Mathematicae* 25 (1935), 503-526, and "Grundzüge des Systemkalkül, Zweiter Teil" ["Foundations of the Calculus of Systems"]. *Fundamenta Mathematicae* 26 (1936), 283-301, in *Logic, Semantics, Metamathematics: Papers from 1923-1939,* 2d ed., trans. J. H. Woodger, ed., with intro. John Corcoran (Indianapolis: Hackett Publishing Co., [1956] 1983), 342-383.

[25]Willard Van Orman Quine, *Mathematical Logic*, rev. ed. (Cambridge, Mass.: Harvard University Press, [1940] 1955), 89.

> Rule of truth-functional inference (TF): To any line or lines we may subjoin, as a new line, any schema which is truth-functionally implied by the given line or by the conjunction of the given lines.[26]

This indefinitely large set of inference rules is also seen a few years later in Suppes' system:

> Rule T: We may introduce a sentence S in a derivation if there are preceding sentences in the derivation such that their conjunction tautologically implies S.[27]

One difficulty with the Quine and the Suppes approach to inference rules is that it is far too strong. For example, Suppes' Rule T justifies writing down the conclusion of *any* valid argument immediately after stating the premises simply by citing Rule T. After all, the premises are written down by Suppes' Rule P and in the case of a valid argument, their conjunction tautologically implies the conclusion. Thus, Suppes' Rule T seems to make every derivation a "one-liner." Hausman provides a very useful discussion of this problem.[28]

In contrast, PARC's Rules A and R are more plausible for a system of natural deduction because they make reference to "elementary" argument forms and "elementary" logical equivalences. Elementary forms and equivalences presumably will include modus ponens, modus tollens, disjunctive syllogism, hypothetical syllogism, double negation, contraposition, implication, De Morgan's Theorem, and the like.

While PARC Rules A and R merely recommend Copi-style rule sets, and the term "elementary" is admittedly vague, the rules have the pedagogical advantage of allowing the instructor to select the desired set of "elementary" rules. There is a minor caveat here of course, since the instructor will need to verify the completeness of the rule set by determining that omitted rules are provable from the remaining rules. It is worth noting that PARC's Rule A and Rule R have the same sort of indefiniteness here as the later Quine and Suppes systems. This distinguishes PARC from Copi-style fixed-size inference and equivalence rule sets. In implementing PARC's Rule A and Rule R, my practice has been to use the implication and equivalence rule sets in either the fifth edition of Copi's *Symbolic Logic* or the most recent edition of *Logic and Philosophy*.

IV. Conditionalization and PARC Rule C

> PARC Rule C: A formula $P \supset Q$ may be written down as a line of a deduction if both (a) there are previous lines of the deduction P, and Q, such that P was written down by Rule P; and (b) all numerals prefixed to Q except the numeral prefixed to P are prefixed to $P \supset Q$.

Conditionalization—known to mathematicians as the "deduction theorem" —is the rule that allows the temporary assumption of a sentence, p, the subsequent derivation of a sentence, q,

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[26]Quine, *Methods of Logic*, 157.

[27]Suppes, *Introduction to Logic*, 28. Suppes observes in a footnote that this rule allows "the conjunction of any finite number of sentences, not just two. Thus we might have $((P \rightarrow Q) \& (Q \rightarrow R)) \& (R \rightarrow S)$, which tautologically implies $P \rightarrow S$"—hypothetical syllogism in this case.

[28]Hausman, *Logic and Philosophy*, 369-372.

and the final, assumption-free assertion of the conditional p ⊃ q.[29] In int-elim systems, conditionalization is known as “conditional introduction” or “horseshoe introduction.”

Most crucially, the rule of conditionalization is, in Quine’s words, “the crux of natural deduction.”[30] As noted above, natural deduction was created—independently by Jaśkowski and Gentzen—out of the desire to replace the axiomatic style of logic found in Whitehead and Russell and in Hilbert with the style of reasoning ordinarily used by logicians and mathematicians, viz., what follows logically if we *assume* that such-and-such is true? This pattern of reasoning certainly dates to antiquity but without any formal statement or even recognition of its centrality in “natural” deductive reasoning.

In the PARC system, the conditionalization Rule C is tied directly to Rule P. This is because of the suppositional character of natural deduction. When used in a subproof, PARC’s Rule P allows the assumption of a temporary premise, but that assumption must be discharged before the last line of the proof. This requires a rule—Rule C—that justifies the creation of a conditional in which the temporary assumption can become the antecedent of the conditional with the statement derived with the aid of the assumption becoming the consequent.[31] As I tell my students—again and again, it seems—“If you assume that P is true and derive Q as a result, then you are entitled to express that fact with the conditional ‘If P, then Q’. Whether P is actually true or not is irrelevant and hence the assumption that it is true may be abandoned.”

Here is a simple example[32] of a conditional proof in the PARC system:

	(1)	A ⊃ B	Rule P	
	(2)	C ⊃ D	Rule P	/∴ (A v C) ⊃ (B v D)
	(3)	A v C	Rule P	
(1,2,3)	4.	B v D	Rule A, 1, 2, 3, Constructive Dilemma	
(1,2)	5.	(A v C) ⊃ (B v D)	Rule C, 3, 4	

Conditional proofs in first-order logic fall into three basic groups. The first sort of proof involves arguments with conditional conclusions or other conditional statements in which the antecedent of the conclusion is assumed in a subproof and the consequent of the conclusion is derived. The target conditional is thus demonstrated and the assumed antecedent is discharged.

[29]The Conditionalization rule in natural deduction is a version of Tarski’s statement of the deduction theorem (Axiom 8 in his system). See Alfred Tarski, “Über einige fundamentale Begriffe der Metamathematik” [“On Some Fundamental Concepts of Metamathematics”], *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), 22-29, in *Logic, Semantics, Metamathematics: Papers from 1923-1939*, 2d ed., trans. J. H. Woodger, ed., with intro. John Corcoran (Indianapolis: Hackett Publishing Co., [1956] 1983), 32. Tarski states in footnote 24 of “Concepts of Metamathematics” that he established the deduction theorem “as far back as 1921.” Alonzo Church disputes Tarski’s claim, stating that “[t]he idea of the deduction theorem and the first proof of it for a particular system must be credited to Jacques Herbrand. Its formulation as a general methodological principle for logistic systems is due to Tarski. . .” See his *Introduction to Mathematical Logic* (Princeton, N. J.: Princeton University Press, [1944, 1956] 1996), 164.

[30]Quine, *Methods of Logic*, 166. See Pelletier, “Brief History,” for a history of the rule of conditionalization.

[31]It is worth noting that from a pedagogical perspective, int-elim systems obscure this connection between making an assumption and discharging it through conditionalization. After all, the conditional introduction rule, ⊃I, is just another connective introduction rule, seemingly on a par with, say, &I (conjunction).

[32]The argument is from *Logic and Philosophy*, 130, repeating a challenging nonconditional exercise from a previous chapter.

The second sort of conditional proof involves indirect proofs, i.e., proofs that assume the negation of the conclusion and seek to derive a contradiction. Once both some statement and its negation have been derived, the assumption is shown to be false, and its negation, the conclusion, is demonstrated.

Finally, proofs in predicate logic that utilize Existential Instantiation or Universal Generalization implicitly involve subordinate proofs that must be included in the category of conditional proofs.

Copi's third edition of *Symbolic Logic* seems to be the source of presenting the conditional underpinnings of all three sorts of subproofs in symbolic logic textbooks. However, Copi then presents derived rules for IP (indirect proof) and EI (existential instantiation) based on conditional proof. Most current textbooks present similar derived rules, but very few demonstrate the underlying conditional reasoning. The PARC system is distinctive among natural deduction systems presented in contemporary textbooks in that all three sorts of subproofs make the conditional reasoning explicit. While the cost is sometimes a slightly longer proof—two or three lines--the pedagogical advantage is that the student again and again sees the common logic that underlies proofs with conditional conclusions, *reductio* proofs, and predicate logic proofs with existential lines.

Beyond Copi, PARC's other contemporaries—especially Suppes, Mates, and Pollock—utilize parts of this common logic, but only PARC requires the student to treat all subproofs as species of conditional proof.[33] The easiest way to understand the first part of this point is to note that PARC has neither a "Law of Absurdity" as an inferential rule to assist indirect proofs, nor an "Indirect Proof" rule. (And, as we'll see below, PARC doe not have an Existential Instantiation quantification rule (existential quantifier elimination (∃E) in int-elim systems) which implicitly requires a conditional subproof.

Schematically, a PARC indirect proof looks like this:

	(1)	p	Rule P /∴ q
	(2)	~q	Rule P
		.	
(1 and/or 2)	m.	r	
		.	
(1 and/or 2)	n.	~r	
(1 and/or 2)	n+1.	r v q	Rule A, m, Addition
(1,2)	n+2.	q	Rule A, n, n+1, Disjunctive Syllogism
(1)	n+3.	~q ⊃ q	Rule C, 2, n+2
(1)	n+4.	~~q v q	Rule R, n+3, Implication
(1)	n+5.	q v q	Rule R, n+4, Double Negation
(1)	n+6.	q	Rule R, n+5, Tautology

I express the indirect proof strategy in PARC to my students as follows:

1. Given a conclusion Z, assume the negation of the conclusion ~Z using Rule P.

[33] *Logic and Philosophy* is one of the few current textbooks that explicitly discusses the conditional basis of an Indirect Proof inference rule (see 139-140).

2. Derive some statement K on one line of your derivation (possibly the conclusion itself, possible a given premise).
3. On a later line, derive $\sim K$ (or vice versa). This completes the contradiction that is the key to a *reductio ad absurdum* argument.
4. Add the conclusion Z to K ($K \vee Z$), and then use the other half of the contradiction, $\sim K$, to derive Z by Disjunctive Syllogism.
5. Write the line $\sim Z \supset Z$, using Rule C. This discharges the assumption.
6. Derive the conclusion Z by Implication, Double Negation, and Tautology.

Here is an example of an indirect proof in PARC:

	(1)	$\sim(B \supset A)$	Rule P	
	(2)	$\sim G \supset A$	Rule P	$/\therefore G$
	(3)	$\sim G$	Rule P	
(2,3)	4.	A	Rule A, 2, 3, Modus Ponens	
(1)	5.	$\sim(\sim B \vee A)$	Rule R, 1, Implication	
(1)	6.	$\sim\sim B \bullet \sim A$	Rule R, 5, De Morgan	
(1)	7.	$\sim A$	Rule A, 6, Simplification	
(2,3)	8.	$A \vee G$	Rule A, 4, Addition	
(1,2,3)	9.	G	Rule A, 7, 8, Disjunctive Syllogism	
(1,2)	10.	$\sim G \supset G$	Rule C, 3, 9	
(1,2)	11.	$\sim\sim G \vee G$	Rule R, 10, Implication	
(1,2)	12.	$G \vee G$	Rule R, 11, Double Negation	
(1,2)	13.	G	Rule R, 12, Tautology	

While we know that an Indirect Proof inference rule is a "short-cut" rule, derived from the conditional proof rule, some textbooks do not place a pedagogical emphasis on this relationship and others don't even mention it. Those that observe the connection may quickly note it and then present the Indirect Proof rule simply because it is "a shorter way of proving what could have been proved by conditional proof."[34] Unhappily, the most obscure approach to *reductio* proofs is found in int-elim systems where the entire *reductio* process—Negation Introduction and Negation Elimination—employs a subproof that appears to the student eye to be totally unrelated to a Conditional Introduction subproof.[35]

V. PARC Quantification Rules

Fm represents any formula in which the variable-type *m* occurs free, and when used in the same context, *Fm* and *Fn* will represent formulae that are just alike except that *Fn* has free tokens of *n* everywhere that *Fm* has free tokens of *m*. It should be noted that *Fn* may have free tokens of *n* over and above the ones that correspond to free tokens of *m* in *Fm*.

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[34]Hausman, et al., *Logic and Philosophy*, 140.

[35]For example, see Karel Lambert and Bas C. van Fraassen, *Derivation and Counterexample: An Introduction to Philosophical Logic* (Belmont, Calif.: Dickenson Publishing Co., 1972), 44 ff.

Rules of Inference:

- **Universal Instantiation (UI):**

  $(\forall m)Fm$

  Fn

- **Existential Generalization (EG):**

  Fn

  $(\exists m)Fm$

- **Universal Generalization (UG):**

  Fn

  $(\forall m)Fm$

  UG Restriction: ***n*** **occurs free neither in any premise of the line** ***(∀m)Fm*** **nor in the line** ***(∀m)Fm*** **itself.**

- **Existential Instantiation (EI):**

  No EI rule of inference in PARC

  Note on EI: Although there is no EI rule, one can make an EI-like move in the following way. Given a line *(∃m)Fm*, you may write down *Fn* by Rule P. Make whatever use of *Fn* you wish, and then remove the prefixed numeral by Rule C. You will then have a line of the form *Fn ⊃ A*. (Warning: make sure that *A* does not contain a free occurrence of *n*.) Now, you may apply UG to the line to get *(∀m)(Fm ⊃ A)*. Using the prenex normal form "prime" equivalence rule (PNF') below, you may write down the next line of the proof as *(∃m)Fm ⊃ A*. You may then conclude *A* by *modus ponens* with the original line *(∃m)Fm*.

Rules of Equivalence:

- **PNF'** ("PNF prime"): $(\forall m)(Fm \supset A) \equiv [(\exists m)Fm \supset A]$ **Restriction: A contains no free occurrence of** ***m***.[36]

- **Quantifier Negation (QN):**

  $(\forall m)Fm \equiv \sim(\exists m)\sim Fm$ $\sim(\forall m)Fm \equiv (\exists m)\sim Fm$
  $(\forall m)\sim Fm \equiv \sim(\exists m)Fm$ $\sim(\forall m)\sim Fm \equiv (\exists m)Fm$

Identity:

Fm	Fm		
$n = m$	$\sim Fn$	$n = m$	p
--------------	---------	---------	---------
Fn	$\sim(n = m)$	$m = n$	$m = m$

[36]This common equivalence in first-order logic can be easily proved in PARC. It is metatheorem 161 in Quine's system in *Mathematical Logic*, 109. John Pollock also uses this quantificational equivalence in lieu of an existential instantiation inference rule in *An Introduction to Symbolic Logic*, 135. The equivalence is Theorem 21 in Copi's RS_1 first-order system, *Symbolic Logic*, 287.

The PARC rules for predicate logic are distinctive—though not unique—because there is no rule for Existential Instantiation, per se. Since in monadic predicate logic, the UI, UG, and EG are very simple, I will focus on arguments that would normally require an EI rule application. The strategy stated in the rule set above is simple: just assume an "instantiation" of an existential statement with Rule P and proceed as described. Here is an example:

	(1)	$(\exists x)Fx$	Rule P $/\therefore$ $(\exists x)(Fx \vee Gx)$
	(2)	Fy	Rule P
(2)	3.	$Fy \vee Gy$	Rule A, 2, Addition
(2)	4.	$(\exists x)(Fx \vee Gx)$	3, Existential Generalization (EG)
(0)	5.	$Fy \supset (\exists x)(Fx \vee Gx)$	Rule C, 2, 4
(0)	6.	$(\forall x)[Fx \supset (\exists x)(Fx \vee Gx)]$	5, Universal Generalization (UG)
(0)	7.	$(\exists x)Fx \supset (\exists x)(Fx \vee Gx)$	6, PNF' Equivalence
(1)	8.	$(\exists x)(Fx \vee Gx)$	Rule A, 1, 7, Modus Ponens

The PARC method for dealing with an existential statement via a conditional subproof is first presented by Copi in the third edition of *Symbolic Logic*.[37] Copi presents the EI subproof schematically, using the the same equivalence as PARC's prenex normal form "prime" rule above.

E: $(\nu)(\Phi\nu \supset p) \equiv [(\exists\mu)\Phi\mu \supset p]$, where ν occurs free in $\Phi\nu$ at all and only those places that μ occurs free in $\Phi\mu$, and where p contains no free occurrence of the variable ν (ibid.).

i.	$(\exists\mu)\Phi\mu$	
	.	
	.	
	.	
j.	$\Phi\nu$	
	.	
	.	
	.	
k.	p	
k+1.	$\Phi\nu \supset p$	j-k, Conditional Proof
k+2.	$(\nu)(\Phi\nu \supset p)$	k+i, Universal Generalization
k+3.	$(\exists\mu)\Phi\mu \supset p$	k+2, Equivalence (E)
k+4.	p	k+3, i, Modus Ponens

Copi acknowledges Leblanc as the source of this approach.[38] EI is then advanced as a derived rule on the basis of the above schema as an "informal justification."[39] The explicit subproof without an EI rule was later adopted by PARC and then by Pollock[40]. In terms of a published

[37]Copi, *Symbolic Logic*, 3d ed., 112.
[38]Ibid., 111.
[39]Ibid., 113.
[40]Pollock, *Introduction to Logic*.

system, Pollock's quantification rules are anticipated by PARC. As in indirect proofs, the inherent conditional proof that underlies EI is made explicit in PARC.

Polyadic Predicate Logic in PARC

There are no surprises here. Of course, like EI, UG in most systems has multiple restrictions. In contrast, one can see that PARC's UG restrictions are minimal. This makes polyadic predicate logic much easier for students. Here is an example that uses UG, UI, EG, as well as handling existential lines.

	(1)	$(\forall x)(\forall y)Lxy$	Rule P
	(2)	$(\exists x)(\forall y)(Lxy \supset Gxy)$	Rule P /∴ $(\exists x)(\forall y)Gxy$
	(3)	$(\forall y)(Lzy \supset Gzy)$	Rule P
(3)	4.	$Lzw \supset Gzw$	3, Universal Instantiation (UI)
(1)	5.	$(\forall y)Lzy$	1, Universal Instantiation (UI)
(1)	6.	Lzw	5, Universal Instantiation (UI)
(1,3)	7.	Gzw	Rule A, 4, 6, Modus Ponens
(1,3)	8.	$(\forall y)Gzy$	7, Universal Generalization (UG)
(1,3)	9.	$(\exists x)(\forall y)Gxy$	8, Existential Generalization (EG)
(1)	10.	$(\forall y)(Lzy \supset Gzy) \supset (\exists x)(\forall y)Gxy$	Rule C, 3, 9
(1)	11.	$(\forall x)[(\forall y)(Lxy \supset Gxy) \supset (\exists x)(\forall y)Gxy]$	10, Universal Generalization (UG)
(1)	12.	$(\exists x)(\forall y)(Lxy \supset Gxy) \supset (\exists x)(\forall y)Gxy$	11, PNF' Equivalence
(1,2)	13.	$(\exists x)(\forall y)Gxy$	Rule A, 2, 12, Modus Ponens

Conclusion

While much of the PARC system of natural deduction is not original, it is unique in combining what I regard as the best-of-all-worlds in first-order logic pedagogy. It tracks premise dependencies, it uses a "natural" set of inference and equivalence rules, and it requires the student to learn—at a deep level—the underlying conditional nature of indirect proofs and quantificational proofs that require an EI-like decomposition. While PARC's rules of sentential logic are commonplace—and easily revised—its quantificational rules are a significant advantage to the student when faced with complex sets of restrictions. And perhaps most significantly, the PARC system can be easily overlaid on top of Copi-style systems that are predominant in current textbooks.[41]

[41]This paper reflects the content of my workshop, "The PARC System of Natural Deduction," conducted at the American Association of Philosophy Teachers 20th International Workshop-Conference on Teaching Philosophy, College of Saint Benedict and Saint John's University, Collegeville, Minnesota, July 31, 2014. I thank the editors for the opportunity to present the PARC system in this volume.

Acknowledgements

I am indebted to Dr. Ralph Slaght, Hogg Professor Emeritus of Philosophy, Lafayette College, for my exposure to the PARC system of natural deduction. I learned this system from Professor Slaght as an undergraduate at Lafayette in 1969. PARC has never been published as far as I can determine and the precise origin of the system remains unknown to me. The above conjectures regarding its sources are my own.

References

Anellis, Irving H. "Forty Years of 'Unnatural' Natural Deduction and Quantification: A History of First-Order Systems of Natural Deduction from Gentzen to Copi." *Modern Logic* 2 (1991), no. 2: 113–152.

Church, Alonzo. *Introduction to Mathematical Logic*. Princeton, N. J.: Princeton University Press, [1944, 1956] 1996.

Copi, Irving M. *Symbolic Logic*. New York: Macmillan Publishing Co, 1954.

Copi, Irving M. *Symbolic Logic*. 3d ed. New York: Macmillan Publishing Co, 1967.

Copi, Irving M. *Symbolic Logic*. 5th ed. New York: Macmillan Publishing Co, 1979.

Fitch, Frederick. *Symbolic Logic: An Introduction*. New York: The Ronald Press, 1952.

Gentzen, Gerhard. "Untersuchungen über das Logische Schließen." ["Investigations into Logical Deduction"]. *Mathematische Zeitschrift* 39 (1934/5): 176-210, 405-431. In *The Collected Works of Gerhard Gentzen*, translated by M. E. Szabo. Amsterdam: North-Holland Publishing Co., 1969, 68-131.

Hausman, Alan, Howard Kahane, and Paul Tidman. *Logic and Philosophy: A Modern Introduction.* 12th ed. Boston: Wadsworth, Cengage Learning, 2013.

Jaśkowski, Stanisław. "On the Rules of Suppositions in Formal Logic." *Studia Logica* 1 (1934), 5-32. Reprinted in *Polish Logic, 1920—1939*, edited by Storrs McCall. London: Oxford University Press, 1967, 232-258.

Jeffrey, Richard C. *Formal Logic: Its Scope and Limits*. New York: McGraw-Hill, 1967.

Lambert, Karel and Bas C. van Fraassen. *Derivation and Counterexample: An Introduction to Philosophical Logic*. Belmont, Calif.: Dickenson Publishing Co., 1972.

Leblanc, Hugues and William A. Wisdom. *Deductive Logic*. 3d ed. Englewood Cliffs, N.J.: Prentice-Hall, [1972, 1976] 1993.

Lemmon, E. J. *Beginning Logic*. London: Thomas Nelson and Sons, 1965.

McCall, Storrs, ed. *Polish Logic, 1920-1939*. With an introduction by Tadeusz Kotarbiński. Papers by Ajdukiewicz, et al. Translated by B. Gruchman, et al. London: Oxford University Press, 1967.

Mates, Benson. *Elementary Logic*. New York: Oxford University Press, 1965.

Pelletier, Francis Jeffry. "A Brief History of Natural Deduction." *History and Philosophy of Logic* 20 (1999): 1-31.

Pelletier, Francis Jeffry. "A History of Natural Deduction and Elementary Logic Textbooks." Paper presented to the Society for Exact Philosophy, 1999. Accessed November 1, 2014. http://www.sfu.ca/~jeffpell/papers/pelletierNDtexts.pdf.

Pollock, John F. *An Introduction to Symbolic Logic*. New York: Holt, Rinehart and Winston, 1969.

Quine, Willard Van Orman. *Mathematical Logic*. Rev. ed. Cambridge, Mass.: Harvard University Press, [1940] 1955.

Quine, Willard Van Orman. *Methods of Logic*. Rev. ed. New York: Holt, Rinehart and Winston, [1950] 1959.

Smullyan, Raymond M. *First-Order Logic*. Corrected ed. New York: Dover Publications, [1968] 1995.

Suppes, Patrick. *Introduction to Logic*. New York: Van Nostrand Reinhold Co., 1957.

Szabo, M. E., ed. *The Collected Papers of Gerhard Gentzen*. Amsterdam: North-Holland Publishing Co., 1969.

Tarski, Alfred. "Über einige fundamentale Begriffe der Metamathematik" ["On Some Fundamental Concepts of Metamathematics"]. *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), 22-29. In *Logic, Semantics, Metamathematics: Papers from 1923-1939*. 2d ed. Translated by J. H. Woodger. Edited, with an introduction by John Corcoran. Indianapolis: Hackett Publishing Co., [1956] 1983, 30-37. Citations are to the translated edition.

Tarski, Alfred. "Grundzüge des Systemenkalkül, Erster Teil." *Fundamenta Mathematicae* 25 (1935), 503-526, and "Grundzüge des Systemenkalkül, Zweiter Teil" ["Foundations of the Calculus of Systems"]. *Fundamenta Mathematicae* 26 (1936), 283-301. In *Logic, Semantics, Metamathematics: Papers from 1923-1939*. 2d ed. Translated by J. H.

Woodger. Edited, with an introduction by John Corcoran. Indianapolis: Hackett Publishing Co., [1956] 1983, 342-383. Citations are to the translated edition.

Tarski, Alfred. *Logic, Semantics, Metamathematics: Papers from 1923-1939*. 2d ed. Translated by J. H. Woodger. Edited, with an introduction by John Corcoran. Indianapolis: Hackett Publishing Co., [1956] 1983.

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