The PARC System of Natural Deduction

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Workshop Overview

1. Brief review of natural deduction
2. Presentation of PARC system
3. Participants write and discuss PARC proofs in sentential logic
4. Participants write and discuss PARC proofs in quantificational logic
5. Discussion of PARC advantages and disadvantages

Approach is informal; 2-5 are spiraled.
Deductive Systems

There are many kinds of deductive systems, but all begin with a propositional or sentential calculus, and a first-order quantificational or predicate calculus that allows proofs of conclusions involving individuals, their properties, and the words, “all,” “some,” and “none.”
Types of Deductive Systems

- Axiomatic
- Natural Deduction
- Sequent Calculus
Natural Deduction

- Natural deduction arose in the 1920s as an alternative to Frege-Russell-Hilbert axiomatic deductive systems.
- Axiomatic systems have a relatively large number of axioms and a small number of inference rules, sometimes just one, such as modus ponens.
- Natural deduction systems are organized in the opposite way with several inference rules and few, or more commonly, no axioms at all (called deductive zero order systems).
- Every line of a natural deduction proof is a valid formula (makes EI semantically problematic).
Co-creators of Natural Deduction

- Stanisław Jaśkowski (1934)
- Gerhard Gentzen (1934/5)
1926 -- Jan Łukasiewicz conducts a seminar focusing on the fact that mathematicians don’t use the axiomatic method in their everyday proofs, but rather use assumptions and inference rules (Woleński 110).


Gerhard Gentzen (Göttingen)

- Gentzen, Paul Bernays’ student, develops a natural deduction system independently of Jaśkowski’s work at Warsaw.
- 1932 – Gentzen publishes a counterexample to Paul Hertz’ 1929 Satzsysteme that requires no axioms.
Gentzen and “Natural” Deduction

- The term “natural deduction” is introduced by Gentzen:

  My starting point was this: The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return.

  In contrast, I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a ‘calculus of natural deduction’ . . . (Gentzen 1934, 68)
The pedagogy of most contemporary first-order logic textbooks is firmly rooted in natural deduction.

W. V. Quine is the most visible early adopter of natural deduction in his logic courses at Harvard in the late 1930s, employing one of Jaśkowski’s methods.
Jaśkowski’s Two Methods

- Natural deduction focuses on making assumptions, so assumption scope has to be carefully managed.
- Quine elected Jaśkowski’s “bookkeeping” method of numerical annotations to proof lines (Quine 1950).
- Jaśkowski’s graphical method of enclosing subproofs in rectangles was adopted by Frederick Fitch (1952).
- Gentzen’s sequent calculus is rarely seen in logic textbooks.
The Origin of PARC

- A “bookkeeping” system somewhat similar to Jaśkowski’s was introduced by Patrick Suppes (1957).
- The Suppes system tracks assumptions by listing premise numbers, i.e., the line numbers of those assumptions which logically entail the given line.
- My view is that the PARC system is based on the Suppes system.
The PARC system of deduction is thus not a totally new system of natural deduction. PARC provides four metarules that clarify deductive proofs and make more explicit certain types of proofs such as conditional and indirect (reductio) proofs, and quantificational proofs.
Lineage of PARC’s Assumption Rule

Suppes (1957): Rule P. We may introduce a premise at any point in a derivation.

Mates (1965): Rule P (Introduction of premises) Any sentence may be entered on a line, with the line number taken as the only premise-number.

Lemmon (1965): Rule of Assumptions (A): Any proposition may be introduced at any stage of a proof. We write to the left the number of the line itself.

PARC (1967?): Rule P: Any formula F may be written down as the \( n \)th line of a deduction (derivation) if the numeral \( n \) is appended to it.

Pollock (1969): Rule P: Premise Introduction: Any formula can be written on any line of a PC derivation provided that we let the premise number be the line number of that line.
PARC Rule of Tautological Implication

**Rule A:** A formula $F$ may be written down as a line of a deduction if both
(a) there are previous lines of the deduction from which $F$ follows by some elementary argument form (e.g., Copi’s Rules of Inference), and
(b) each numeral appended to any of these previous lines is appended to $F$.

**Suppes:** Rule T: We may introduce a sentence $S$ in a derivation if there are preceding sentences in the derivation such that their conjunction tautologically implies $S$. 
**Rule R:** A formula F may be written down as a line of a deduction if both
(a) there is a previous line L of the deduction and a pair of elementary logically equivalent formulae G and H such that G is a formula that occurs in L and it is possible to transform L into F by putting H in place of one occurrence of G, and
(b) all numerals appended to L are appended to F.

Suppes: see previous Rule T. Suppes does not distinguish between tautological implication rules and tautological replacement rules.
Rule P Example

- Rule P is used in *every* proof to justify entering a premise—a premise is just another assumption, of course.

- Here’s an example of PARC’s rule citation and the prefixed numerals (without any subproofs):

  1. A ⊃ B  
  2. B ⊃ C  
  3. (A ⊃ C) ⊃ (B ⊃ D)  
  4. (A ⊃ D) ⊃ E  
  5. A ⊃ C  
  6. B ⊃ D  
  7. A ⊃ D  
  8. E  

  - (1)  
  - (2)  
  - (3)  
  - (4)  
  - (1,2)  
  - (1,2,3)  
  - (1,2,3)  
  - (1,2,3,4)

  - Rule P  
  - Rule P  
  - Rule P  
  - Rule P  
  - Rule P / ∴ E  
  - Rule A, 1, 2, Hypoth. Syllog.  
  - Rule A, 3, 5, Modus Ponens  
  - Rule A, 1, 6, Hypoth. Syllog.  
  - Rule A, 4, 7, Modus Ponens
The rule of conditionalization is also known as the “Deduction Theorem.”

According to Quine, the rule of conditionalization is “the crux of natural deduction” (Quine 1950, 166).

Conditionalization is a version of Tarski’s (1930, 32) deduction theorem (Axiom 8 in his system):

If $X \subseteq S$, $y \in S$, $z \in S$ and $z \in \text{Cn}(X + \{y\})$, then $c(y, z) \in \text{Cn}(X)$

$S$ is the set of all sentences, $\text{Cn}(X)$ is the set of sentences that are the consequences of the set $X$, and $c(x, y)$ indicates material implication where $x$ is the antecedent and $y$ is the consequence. Tarski states in a footnote that he established the deduction theorem “as far back as 1921” (Tarski 1930, 32 fn †).
Alonzo Church states that “[t]he idea of the deduction theorem and the first proof of it for a particular system must be credited to Jacques Herbrand. Its formulation as a general methodological principle for logistic systems is due to Tarski. . . . The idea of using the deduction theorem as a primitive rule of inference in formulations of the propositional calculus or the functional calculus is due independently to Jaśkowski and Gentzen” (Church 1996, 164).
PARC Rule of Conditionalization

**Rule C:** A formula $P \supset Q$ may be written down as a line of a deduction if both
(a) there are previous lines of the deduction $P$, and $Q$, such that $P$ was written down by Rule $P$, and
(b) all numerals appended to $Q$ except the numeral appended to $P$ are appended to $P \supset Q$. 
Conditionalization in Brief

- Basically Rule C says that if you assume that some statement S is true and with that assumption you can derive another statement T, then you’re entitled to say just that, viz., $S \Rightarrow T$

- Related to the Rule of Exportation:

$$[(p \land q) \Rightarrow r] \equiv [(p \Rightarrow (q \Rightarrow r))]$$
# Rule C in a Conditional Conclusion

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<td>(1)</td>
<td>A ⊃ B</td>
<td>Rule P</td>
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<td>(2)</td>
<td>C ⊃ D</td>
<td>Rule P / ∴ (A v C) ⊃ (B v D)</td>
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<td>(3)</td>
<td>A v C</td>
<td>Rule P</td>
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<td>(1,2)</td>
<td>(A v C) ⊃ (B v D)</td>
<td>Rule C, 3, 4</td>
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- Note that the assumption made in line 3 is discharged when Rule C is applied in line 5.
- Rule C is also used in indirect proofs (proof by *reductio ad absurdum*)
Rule C in Reductio ad Absurdum Proofs

- Reductio ad absurdum proofs are conditional proofs.
- Most current logic texts use a derived Indirect Proof inference rule that obscures the role of conditionalization.
- Suppes uses a “Law of Absurdity” to complete indirect proofs, viz.,
  \[ [p \implies (q \land \neg q)] \implies \neg p \]
  That is, if a statement materially implies a contradiction, then the statement is false.
- PARC unpacks all of the details.
Rule C in a PARC Indirect Proof

(1) \( \sim (B \Rightarrow A) \) Rule P
(2) \( \sim G \Rightarrow A \) Rule P \(/.: G\)
(3) \( \sim G \) Rule P
(2,3) 4. \( A \) Rule A, 2, 3, Modus Ponens
(1) 5. \( \sim (\sim B \lor A) \) Rule R, 1, Implication
(1) 6. \( \sim \sim B \& \sim A \) Rule R, 5, De Morgan’s Theorem
(1) 7. \( \sim A \) Rule A, 6, Simplification
(2,3) 8. \( A \lor G \) Rule A, 4, Addition
(1,2,3) 9. \( G \) Rule A, 7, 8, Disjunctive Syllogism
(1,2) 10. \( \sim G \Rightarrow G \) Rule C, 3,9
(1,2) 11. \( \sim \sim G \lor G \) Rule R, 10, Implication
(1,2) 12. \( G \lor G \) Rule R, 11, Double Negation
(1,2) 13. \( G \) Rule R, 12, Tautology
Quantification in Natural Deduction

Quantification rules in natural deduction typically include

1. Universal Generalization (UG or $\forall$ Introduction)
2. Universal Instantiation (UI or $\forall$ Elimination)
3. Existential Generalization (EG or $\exists$ Introduction)
4. Existential Instantiation (EI or $\exists$ Elimination)

1 and 4 are the tricky ones. For example, we don’t want to conclude that everything is blue just because the sky is blue (illicit UG), nor do we want to conclude that my dog is blue just because at least one thing in the world is blue (illicit EI). But EI is more than tricky—it is not a true rule of inference, but rather is a rule of “strategy” (Gupta 1968) for introducing a temporary assumption.
Quantification in Natural Deduction

- This leads to “restrictions” on the application of quantificational rules.
- Restrictions are simple enough to explain to logic students, but they are notoriously troublesome in application.
- Quantification restrictions have a distinct literature in contemporary logic.
- The authors of the logic text I currently use (Logic and Philosophy) have discovered a restriction error that had existed through nine editions!
The PARC quantificational rules have very simple restrictions compared to competing systems. In fact, there are only three quantificational rules in PARC instead of four.
Existential Instantiation (EI)

- The 1960s literature on natural deduction shows extensive difficulties with correctly formulating EI. (See Anellis 1991).
- Gentzen, Quine, Leblanc, and others all understood that EI actually involves adding an assumption to a proof.
- Copi adopts Leblanc’s corrections in the third edition of *Symbolic Logic* (Copi 1967, 111).
- The PARC system explicitly uses Copi’s “informal justification” of EI instead of an EI rule (ibid., 113).
Understanding EI to be a form of subproof is well-understood, but rarely practiced in logic pedagogy.

The subproof approach to EI reveals instantiation for what it really is: given that something or other is funny--(∃x)Fx--we can temporarily assume that the name of that thing is n. That is, we assume that Fn is true and see where that assumption leads us, say, to some statement A.
Existential Instantiation in PARC

- PARC has no EI, instead utilizing a well-known equivalence that PARC calls the “prenex normal form prime” rule:

\[(\forall m)(Fm \supset A) \equiv [(\exists m)Fm \supset A]\]

Restriction: \(A\) contains no free occurrence of \(m\).

- The origin of EI as a subproof using this equivalence is presented by Copi in Symbolic Logic 3\(^{rd}\) ed. (1967).
PARC Proofs with Existential Premises

Given a line \((\exists m)F_m\), you may write down \(F_n\) by Rule P. Make whatever use of \(F_n\) you wish, and then remove the prefixed numeral by Rule C. You will then have a line of the form \(F_n \supset A\). (Warning: make sure that \(A\) does not contain a free occurrence of \(n\).) Now, you may apply UG to the line to get \((\forall m)(F_m \supset A)\). Using the prenex normal form equivalence rule (PNF'), you may write down the next line of the proof as \((\exists m)F_m \supset A\). You may then conclude \(A\) by modus ponens with the original line \((\exists m)F_m\).
Acknowledgments

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- Ralph Slaght, Hogg Professor Emeritus of Philosophy, Lafayette College

I am indebted to Dr. Slaght for my exposure to the PARC system of natural deduction. I learned this system from Professor Slaght as an undergraduate at Lafayette in 1969. The precise origin of the system is remains unknown to me and PARC has never been published as far as I can determine. The above conjectures regarding its source are my own.

Selected References


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